Probabilistic Methods of Deterministic Theorems in Mathematics

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Abstract - Ramsey Theory studies the existence of a specific ordered substructure given a mathematical structure of a certain size.

"The fundamental kind of question Ramsey theory asks is: can one always find order in chaos? If so, how much? Just how large a slice of chaos do we need to be sure to find a particular amount of order in it?" - Imre Leader.

The essence of this branch of mathematics is best described by this question, to find how large must a structure be to observe order of a certain size, and how "order" emerges within the structure. This paper will go through the 2 main results of Ramsey Theory, which are the Ramsey Theorem and the van der Waerden Theorem. The discussion of the theorems will include the computation of the special numbers in each theorem, and their bounds.

Keywords - Ramsey Theory, van der Waerden, Hales-Jewett, graph theory, combinatorics, colouring.

1 INTRODUCTION

This paper studies a branch of mathematics in the field of combinatorics called Ramsey Theory. In general, Ramsey Theory is the mathematical study of combinatorial objects where a certain degree of order appears as the size of the object increases.

This branch of mathematics is named after the British mathematician Frank Plumpton Ramsey, who first discovered this field of mathematics and proved its first result, the Ramsey Theory. The theory has then developed greatly through major contributions from Paul Erdös, who devoted his life as a mathematician, predominantly in the field of number theory and combinatorics. Ramsey Theory now has various application in numerous branches of mathematics, primarily in the field of graph theory and combinatorics.

A simple canonical problem brought up by Erdös that has popularized the Ramsey Theory is called 'The Party Problem', a fundamental application of the Ramsey Theorem. This problem describes a scenario where there are n people at a party, and every 2 people in the party either knows each other or does not know each other. It can then be proved that when $n = 6$, there will be a set of 3 people that either knows one another or are completely strangers to each other. Though simple, this result gives us a glimpse of a situation where order emerges within chaos.

At the heart of Ramsey Theory is the Pigeonhole Principle, a simple result that will be used throughout the proofs of the theorems.

Theorem 1.1 (Pigeonhole Principle). *If there are n pigeons to be fitted into m pigeonholes where $n > m$, then at least one of the pigeonholes must contain 2 pigeons.* [1]

2 RAMSEY'S THEOREM

The Ramsey's Theorem is the result proved by Frank Ramsey that first sparked the idea of Ramsey Theory. The theorem demonstrates how an ordered substructure emerges in the field of graph theory.

Definition 2.1 (Coloring). *If c is an n-coloring of a graph $G(V, E)$ with set of vertices $V(G)$ and set of edges $E(G)$, then c is the mapping $c : E(G) \to [n]$, i.e. every edge in G is mapped into a color $x \in \{1, 2, \dots, n\}$.* [2]

Example 2.1. The following is an example of the different 2-colorings of the complete graph, K_4

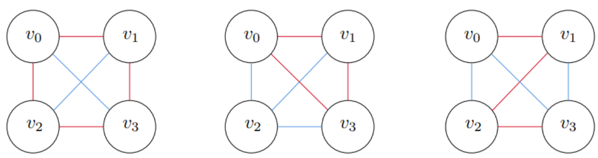


Figure 1.1: Three different 2-colorings of K_4.

Definition 2.2 (Monochromatic Clique). *A clique with n vertices, K_n, is monochromatic if every edge in the clique is assigned the same color.*

Note that the type of graph coloring used in this theory is edge coloring, in the similar way a graph is commonly colored by assigning its vertices to colors in other fields in mathematics. Therefore, a "colored" graph in this paper will refer to edge coloring for the purposes of this theory.

Though a graph can be n-colored, we will deal mainly with the 2-coloring of graphs in this paper, and by convention, red and blue will be used to color the edges.

Theorem 2.1 (Ramsey's Theorem). *Let r and b be integers. Then, there exists a smallest integer n such that for every 2-coloring of the complete graph K_n, there exists either a red monochromatic subgraph K_r or a blue monochromatic subgraph K_b of K_n.*

The proof of this theorem will be provided in the next section.

2.1 RAMSEY NUMBERS

These numbers are the special numbers called Ramsey numbers, denoted by $R(r, b)$. Revisiting the Party Problem introduced earlier, notice that this is actually an application of the Ramsey's Theorem, particularly the Ramsey number $R(3,3)$.

Proof. This problem can be represented as a 2-coloring of the complete graph, K_6. Suppose the 6 people in the party are identified as vertices. Let a red edge denote a pair of people knowing each other and a blue edge denote a pair of people not knowing each other. Pick a random person in the party, say A.

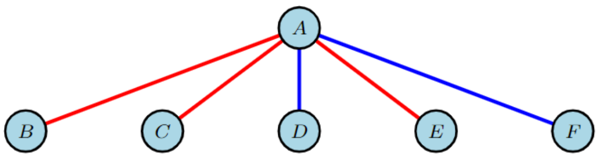


Figure 2.1: Edges connected with A in the clique.

By the generalized Pigeonhole Principle, A must have the same "connection" with at least 3 other people, say B, C, E.

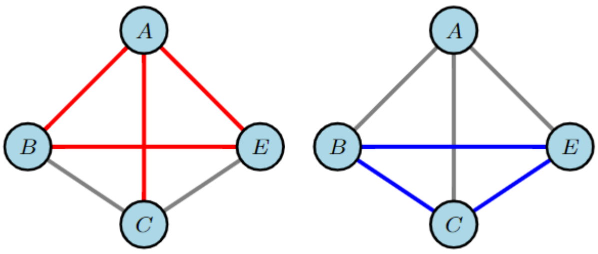


Figure 2.2: Possible edge colorings between B, C, E.

Either 1 of the edges B, C, and E is colored red or all the edges are colored blue. If 1 of the edges is red, then it forms a red complete subgraph of size 3 with A as shown on the left. If all 3 edges are blue, then B, C, E will form a blue complete subgraph of size 3. Therefore, at least 3 people from the party either knows each other or does not know each other.

In the theory of Ramsey numbers, this result only proves that $R(3,3) \leq 6$, as it is yet to be proven that 6 is the least integer satisfying the property.

Theorem 2.2. $R(3,3) = 6$.

To complete this proof, it is enough to prove that $R(3,3) \neq 5$, i.e. $n = 5$ does not satisfy the property. It can be done by showing the negation of the theorem, which is when $n = 5$, there exists a 2-coloring of K_5 such that there does not exist a red complete subgraph K_3 and there does not exist a blue complete subgraph K_3.

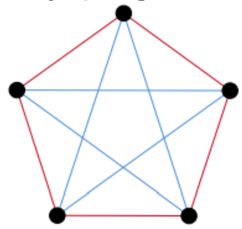


Figure 2.3: A 2-coloring of K_5 without any monochromatic K_3.

Theorem 2.3. *For all $r, b \in \mathbb{N}$, $R(r, b) \leq R(r-1, b) + R(r, b-1)$.*

This inequality is a very powerful recursive bound for Ramsey numbers, used in several Ramsey number bound proofs. This property can be proven by showing that if $n = R(r-1, b) + R(r, b-1)$, then any 2-coloring of K_n will produce a monochromatic K_r or K_b.

Proof. Let $n = R(r-1, b) + R(r, b-1)$. Consider any 2-coloring of K_n. Fix any vertex x of K_n. Let R_x denote all the red edges connected to x and B_x denote all the blue edges connected to x. Therefore, since there are n vertices in K_n, the property $R_x + B_x + 1 = n = R(r-1, b) + R(r, b-1)$ is true.

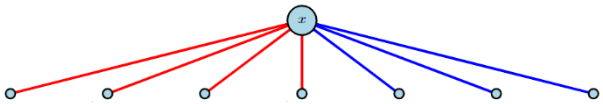


Figure 2.4: Edges connected to x

So, by the Pigeonhole Principle, there is either at least $R(r-1, b)$ red edges or at least $R(r, b-1)$ blue edges.

Case 1: $R_x \geq R(r-1, b)$ and $B_x < R(r, b-1)$. Now, consider the vertices connected with x by a red edge. Then, by the definition of Ramsey numbers, there exists either a red K_{r-1} or a blue K_b. If a red K_{r-1} exists, then it forms a red K_r by being connected to x via a red edge. Therefore, either a monochromatic K_r or K_b will be produced which satisfies our goal.

Case 2: $B_x \geq R(r, b-1)$ and $R_x < R(r-1, b)$. Now, consider the vertices connected with x by a blue edge. By a similar argument as Case 1, a monochromatic K_r or K_b will be produced.

Thus, the proof is complete, and from this result, it can be used to prove the Ramsey's Theorem through induction.

Proof. Recall that the Ramsey's Theorem states the existence of the Ramsey number $R(r, b)$ for all $r, b \in \mathbb{N}$. Then, induction is done on $r + b$, with $r + b = 2$ as the base case. This is only true for $r = b = 1$, and it is clear that $R(1,1) = 1$.

Now, suppose $R(r, b)$ exists for all $r + b < N$ for some positive integer N. Let k and l be positive integers such that $k + l = N$. Since $k + l - 1 < N$, from the assumption earlier, $R(k, l-1)$ and $R(k -$

$1, l)$ exists. From **Theorem 2.3**, it is known that $R(k,l) \leq R(k-1,l) + R(k,l-1)$. Therefore, $R(k,l)$ exists and the inductive step is complete.

2.2 RAMSEY NUMBER VALUES

Currently, Ramsey numbers are a huge point of interest in the Ramsey Theory and only relatively few have been discovered in the world due to its high complexity.

A famous hypothetical problem given by Paul Erdös best describes the difficulty of discovering Ramsey numbers. It says that suppose an evil alien threatens mankind to find the value of $R(5,5)$ or be exterminated. Then, it would be best that we devote all our efforts to compute it with computer and mathematics. However, if the alien asked for the value of $R(6,6)$, then it would be better off to fight the alien off our planet since it is practically impossible to compute the value of $R(6,6)$.

Theorem 2.4. $R(1,k) = 1$ for all $k \in \mathbb{N}$.

It is relatively easy to prove as K_1 only has a single vertex without any edges to be colored.

Theorem 2.5. $R(2,k) = k$ for all $k \in \mathbb{N}$.

Proof. Consider any 2-coloring of K_k. If any of the edges in K_k are colored red, then a red K_2 will be formed. On the other hand, if all the edges are colored blue, then it will be a blue monochromatic K_k. So, $R(2,k) \leq k$.

It can easily be proven that $R(2,k) > k-1$ by coloring all the edges in K_{k-1} blue. It is impossible to form a red monochromatic K_2 or blue K_k.

Theorem 2.6. $R(3,4) = 9$.

Proof. It is known that $R(2,4) = 4$ and $R(3,3) = 6$ from **Theorem 2.2** and **Theorem 2.5**. So, we know that $R(3,4) \leq 10$. However, it can be proven that $R(3,4) \leq 9$. This proof relies on the First Theorem of Graph Theory.

Lemma 2.7. (First Theorem of Graph Theory). *In a graph G, the sum of the degrees of the vertices is equal to twice the number of the edges. Consequently, the number of vertices with odd degree is even.* [3]

Consider any 2-coloring of K_9. Fix any vertex x.

Case 1: x is connected to at least 4 red edges.

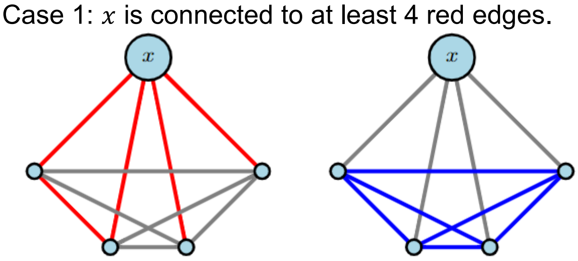


Figure 2.5: Possible formations of red K_3 or blue K_4.

Consider the 4 vertices connected to x via a red edge. Note that $R(2,4) = 4$, so either a red K_2 or a blue K_4 will form within the vertices. If a red K_2 forms, it will then produce a red K_3 with the vertex x.

Case 2: x is connected to at least 6 blue edges. Consider the 6 vertices connected to x via a blue edge. It is known that $R(3,3) = 6$, so by a similar argument as Case 1, either a red K_3 or a blue K_4 will be formed.

Case 3: x is connected to exactly 3 red edges and 5 blue edges. As x is chosen arbitrarily, this applies to every vertex in the graph. Now, consider the red monochromatic subgraph formed in the graph and denote it as G_r. Notice that G_r has 9 vertices and each vertex has a degree of 3. This causes a contradiction as **Lemma 2.7** states that G_r must have an even number of vertices. Therefore, this case cannot exist.

It has been proven that $R(3,4) \leq 9$ by splitting the cases of an arbitrary vertex in K_9. To complete the proof of the theorem, it must be shown that $R(3,4) > 8$, i.e. there exists a 2-coloring of K_8 such that there does not exist a red K_3 and a blue K_4.

We have seen that if any vertex is connected to at least 4 red edges or 6 blue edges, a red K_3 or a blue K_4 will be formed. Therefore, K_8 must be 2-colored in a way that does not contain a vertex with these conditions. It is possible to find a 2-coloring of K_8 where every vertex is connected to 3 red edges and 4 blue edges.

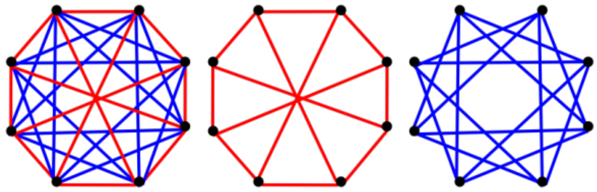


Figure 2.6: A 2-coloring of K_8 that does not contain a red K_3 or a blue K_4.

Therefore, $R(3,4) > 8$ and the proof is complete. [4]

Indeed, the proofs of even the small Ramsey number values are not intuitive. In fact, the only known values of other Ramsey numbers are $R(3,5) = 14, R(3,6) = 18, R(3,7) = 23, R(3,8) = 28, R(3,9) = 36, R(4,4) = 18,$ and $R(4,5) = 25$.[2] The only known Ramsey number value in the case of multi-coloring is $R(3,3,3) = 17$. [4]

2.3 RAMSEY NUMBERS BOUNDS

Since finding exact Ramsey number values is not possible for some numbers, it is only possible to find generalized bounds for Ramsey numbers.

Theorem 2.8. *For all* $r, b \in \mathbb{N}$, $R(r,b) \leq \binom{r+b-2}{r-1}$.[5]

The proof requires the Pascal's Rule as a lemma.

Lemma 2.9. (Pascal's Rule). $\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$

Proof. The proof will be done through double induction on r and b. For the base case $r = b = 2$, $R(2,2) = 2 \leq \binom{2+2-2}{2-1} = 2$. This theorem holds in this case.

Suppose this theorem holds for $R(r-1,b)$ and $R(r,b-1)$. By **Theorem 2.3**, $R(r,b) \leq R(r-1,b) + R(r,b-1) \leq \binom{p+q-3}{p-2} + \binom{p+q-3}{p-1} = \binom{p+q-2}{p-1}$

Theorem 2.10. *$R(k,k) > 2^{\frac{k}{2}}$ for $k \geq 3$.*

We will prove the lower bound of $R(k,k)$ through a probabilistic proof presented by Hung Q. Ngo. The probabilistic proof essentially states that if $n = 2^{\frac{k}{2}}$ and a random 2-coloring of the graph K_n does not produce a monochromatic K_k with probability 1, then there must exist a 2-coloring of K_n without a monochromatic K_k and $R(k,k) > n$.

Lemma 2.11. *If n is an integer such that $\binom{n}{k}2^{1-\binom{k}{2}} < 1$, then $R(k,k) > n$.*

Proof. Consider the 2-coloring of K_n where every edge is colored red or blue with probability of 0.5. Suppose we choose k vertices from the graph randomly and denote it as G_k. There are $\binom{k}{2}$ vertices in it, and the probability of G_k being a monochromatic K_k is $2 * \left(\frac{1}{2}\right)^{\binom{k}{2}} = 2^{1-\binom{k}{2}}$. Note that there are $\binom{n}{k}$ K_k graphs in a K_n graph, therefore, the probability of finding a monochromatic K_k in a 2-coloring of K_n is $\binom{n}{k}2^{1-\binom{k}{2}}$. Therefore, if this probability is not equal to 1, then there exists a 2-coloring of K_n that does not contain a monochromatic K_k, i.e. $R(k,k) > n$.

Next, we use this lemma to prove the lower bound $R(k,k) > 2^{\frac{k}{2}}$ for $k \geq 3$. We note that $\binom{n}{k}2^{1-\binom{k}{2}} = \frac{n!}{k!(n-k)!}2^{1-\frac{k(k-1)}{2}} < \frac{n^k}{k!}\frac{2^{1+\frac{k}{2}}}{2^{\frac{k^2}{2}}}$ as $\frac{n!}{(n-k)!} = n(n-1)\cdots(n-k+1) < n^k$. We continue the proof by substituting $n = 2^{\frac{k}{2}}$ into the lemma proven earlier and proving that the value is less than 1. $\binom{2^{\frac{k}{2}}}{k}2^{1-\binom{k}{2}} < \frac{\left(2^{\frac{k}{2}}\right)^k}{k!}\frac{2^{1+\frac{k}{2}}}{2^{\frac{k^2}{2}}} = \frac{2^{1+\frac{k}{2}}}{k!} < 1$ and this concludes the proof that $R(k,k) > 2^{\frac{k}{2}}$. [6]

2.4 INFINITE RAMSEY'S THEOREM

Previously, we have dealt with the Ramsey's Theorem in a finite point of view where the complete graphs have a finite number of vertices and edges. However, can we expand the Ramsey's Theorem into an infinite scale? Can we see an infinitely large degree of order forming within an infinite structure?

Theorem 2.12. (Infinite Pigeonhole Principle). *If there are infinite pigeons to be fitted into a finite amount of pigeonholes, then at least one of the pigeonholes must contain infinite pigeons.* [1]

The Infinite Pigeonhole Principle is a good example of this and we will see how it relates to the Ramsey's Theorem.

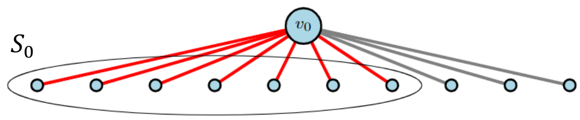
Definition 2.3. *$K_{\mathbb{N}}$ is the complete graph that contains a countably infinite amount of vertices*

We can imagine $K_{\mathbb{N}}$ to have the set $\mathbb{N}$ as its vertices and all subsets of $\mathbb{N}$ with the size 2, i.e. $\{A \subset \mathbb{N} : |A| = 2\}$ as its edges. A 2-coloring of $K_{\mathbb{N}}$ would be the mapping of the set $\{A \subset \mathbb{N} : |A| = 2\}$ into 2 colors.

Theorem 2.13. (Infinite Ramsey's Theorem). *Any 2-coloring of $K_{\mathbb{N}}$ contains a monochromatic countably infinite complete graph.*

Proof. Let c be a 2-coloring of $K_{\mathbb{N}}$. We will fix an arbitrary vertex and select a vertex set that satisfies the conditions. Let v_0 be an arbitrary vertex. v_0 is connected to an infinite amount of edges, but there are only red and blue to color every edge. Therefore, by the Infinite Pigeonhole Principle, v_0 is connected to infinitely many either red or blue edges. Suppose v_0 is connected to infinite red edges.

Figure 2.7: v_0 is connected to infinite red edges.



Denote this infinite set of vertices S_0. Next, we choose an arbitrary vertex v_1 from S_0, and it is also connected to the infinitely many vertices in S_0. Therefore, v_1 must be connected to infinitely many, say, blue edges.

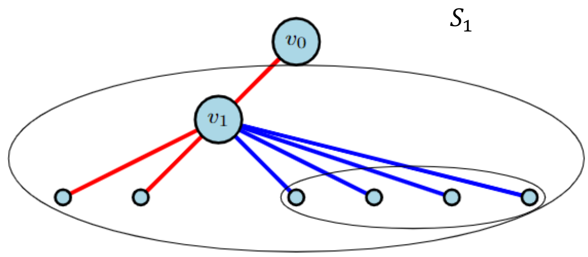


Figure 2.8: v_1 is connected to infinitely many blue edges.

Denote this infinite set of vertices connected to v_1 via a blue edge as S_1. Continue this procedure and pick an arbitrary vertex v_2 from S_1, and notice that v_2 is connected to infinitely many vertices in S_1 by a red or a blue edge. Therefore, there is an infinite set of vertices S_2 in S_1 that are connected to v_2 by, say, red edges.

As we repeat this process, we will obtain the set of vertices $\{v_0, v_1, \cdots, v_n\}$ and the set of infinite vertex sets $\{S_0, S_1, \cdots, S_n\}$ where $\mathbb{N} \supseteq S_0 \supseteq S_1 \supseteq \cdots \supseteq S_n$ and $v_{i+1} \in S_i$ for all i. Suppose we "color" every set S_i by red or blue by the edge color that forms the vertex set, e.g. S_0 is colored red, S_1 is colored blue, etc. Note that $c(v_i, v_{i+1}) = c(v_i, v_{i+2}) = \cdots = c(v_i, v_n)$ for all i. By the Infinite PHP, there must exist an infinite red or blue vertex set, that is $S = \{S_i : c(S_j) = c(S_k)\ \forall j, k\}$. Therefore, every vertex that forms each vertex set in S, or having the same index as the vertex sets, must be connected by the same color. Denote this set of vertices as V. The complete subgraph constructed from V has countably infinite vertices all connected by the same color, thus forming a monochromatic infinite complete subgraph of $K_{\mathbb{N}}$, as desired.

3 VAN DER WAERDEN'S THEOREM

In Ramsey's Theorem, we have looked at how a certain degree of order appears within a mathematical object in the field of graph theory in terms of coloring. In this next part of the Ramsey Theory, we will see how the specific order forms in the field of sequences, particularly on the natural number field. Firstly, the concept of coloring a sequence of numbers have to be introduced.

Definition 3.1. (Coloring). *If c is an n-coloring of a set A, then c is the mapping $c: A \to [1, n]$.*

Example 3.1. The following is an example of the different 2-colorings of the set [1,8].

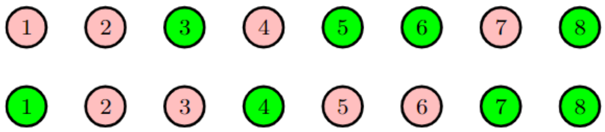


Figure 3.1: Ways to 2-color the set [1,8].

Definition 3.2. (Monochromatic k-Arithmetic Progression). *If an n-coloring of a set contains a monochromatic k-arithmetic progression, or k-AP, then there exist numbers a, b such that $c(a) = c(a+b) = \cdots = c(a+(k-1)b)$.*

To visualize this, in the bottom figure of **Example 3.1**, the 2-coloring of [1,8] contains a monochromatic 3-AP from the terms 1, 4, and 7 that are colored green and form a 3-term arithmetic progression.

Definition 3.3. (Color-focus). *Let c be a k-coloring of the set $[1, n]$. Then, it contains an l-term arithmetic progression r-color-focus at f if there exist r l-term arithmetic progressions $A_1, A_2, \cdots, A_r \subseteq [1, n]$ such that A_i are monochromatic and of different colors for all $i \in [1, r]$, and the $(l+1)^{th}$ term of every set converges to f.* [8]

Example 3.2.

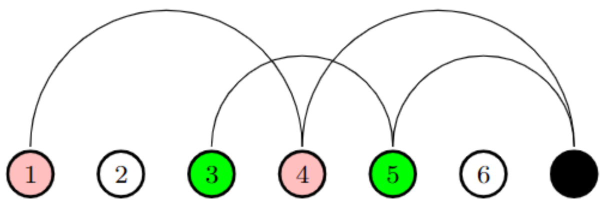


Figure 3.2: The 2-coloring of [1,7] contains a 2-term AP 2-color-focus at 7.

Theorem 3.1. (van der Waerden's Theorem). *Let k, l be any integer. There exists a smallest integer n such that any l-coloring of the set $[1, n]$ contains a monochromatic k-term arithmetic progression.* [9]

These special numbers are called the van der Waerden numbers, denoted as $W(k, l)$.

Example 3.3. $W(2, l) = l + 1$ for all $l \in \mathbb{N}$.

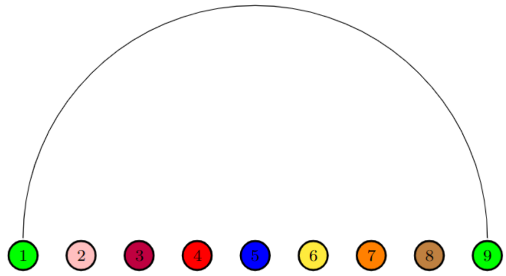


Figure 3.3: $W(2,8) = 9$

If we take the set $[1, l+1]$ and l-color the numbers, there are l colors to color the set $[1, l]$ differently and avoid a monochromatic 2-AP. However, the $(l+1)^{th}$ term will have to be colored using one of the l colors used in the set $[1, l]$, thus creating the monochromatic 2-AP.

Recall. (**The Pigeonhole Principle**). *If there are n pigeons to be fitted into m pigeonholes where $n > m$, then at least one of the pigeonholes must contain 2 pigeons.*

Notice how a similarity exists between this example to the Pigeonhole Principle. If we imagine the numbers in $[1, l+1]$ as the pigeons to be fitted into l colors, then at least 1 color would be used in 2 numbers. This would induce the monochromatic 2-AP as stated.

Example 3.4. $W(3,2)$.

This is a very renowned example of the van der Waerden number. To find the value of $W(3,2)$, we would need to find the set $[1, W(3,2)]$ such that any 2-coloring always induces a 3-term monochromatic AP. There is no systematic way to solving this problem except the brute force to try every possible coloring.

It is possible to color the set [1,7] by $\{1, 2, 3, 4, 5, 6, 7\}$, which does not contain any monochromatic 3-AP, so $W(3,2) > 7$. [1,8] can also be colored $\{1, 2, 3, 4, 5, 6, 7, 8\}$, so $W(3,2) > 8$. Then, is it possible to continue to find these 2-colorings that avoids a monochromatic 3-AP as we increase the size of the set?

If we continue this technique, we would find that it is impossible to find a 2-coloring of [1,9] that avoids a monochromatic 3-AP. Thus, in this case, $W(3,2) = 9$. But, does this number always exist as we increase k and l?

Proof. Perform induction on k. The goal is to proof the existence of $W(k, l)$ for all k, l.

For the base case of $k = 2$, from **Example 3.3,** $W(2, l)$ exists for all l. Suppose that $W(k, l)$ exists for all $k \geq 2$. Now, we have to show that $W(k + 1, l)$ exists

Next, we claim that for any $r \leq l$, there exists the integer $n = W(r, k, l)$ such that any l-coloring of the set $[1, n]$ always induces a monochromatic $(k + 1)$-AP, or a k-AP r-color-focus in the set. We will prove this claim by induction on r.

For the base case of $r = 1$, choose $n = 2W(k, l)$. By the definition of the van der Waerden numbers, any l-coloring of the set $[1, W(k, l)]$ will induce 1 monochromatic k-AP, which will create the 1-color-focus located at the set $[W(k, l) + 1, 2W(k, l)]$. So, $W(1, k, l)$

Suppose that for any $r \leq l$, there exists $n = W(r, k, l)$ where any l-coloring of the set $[1, n]$ induces a monochromatic $(k + 1)$-AP or a k-term AP r-color-focus in the set. Now, we must prove the existence of $W(r + 1, k, l)$.

We begin by choosing $N = 2n * W(k, l^{2n})$. Split the set $[1, N]$ into $W(k, l^{2n})$ intervals of size $2n$. So, $[1, N] = [1,2n] \cup [2n + 1, 4n] \cup \cdots \cup [2n(i - 1) + 1, 2ni] \cup \cdots \cup [2n(W(k, l^{2n}) - 1), 2n * W(k, l^{2n})] = B_1 \cup B_2 \cup \cdots \cup B_{W(k,l^{2n})-1} \cup B_{W(k,l^{2n})}$. Every block B_i is of the same size of $2n$.

Let c be an l-coloring of $[1, N]$. There l^{2n} ways to l-color a set of size $2n$, so every block B_i must be l-colored in one of these l^{2n} ways.

By the definition of the van der Waerden number $W(k, l^{2n})$, any l^{2n}-coloring of the set $[1, W(k, l^{2n})]$ will produce a monochromatic k-term AP. Now, imagine the set $[1, N]$ to be a sequence of blocks $\{B_1, B_2, \cdots, B_{W(k,l^{2n})}\}$ and every l-coloring of a block B_i represents a color. This becomes an l^{2n}-coloring of the set $[1, W(k, l^{2n})]$.

So, by definition, this induces a monochromatic k-term AP, that is it contains k blocks of an identical l-coloring that forms an arithmetic progression. Denote these blocks as $B_a, B_{a+d}, \cdots, B_{a+(k-1)d}$.

Note that every block is of the size $[1,2n]$, and by definition of $n = W(r, k, l)$, each block contains a monochromatic $(k + 1)$-AP or an r-color-focus. If any contains a monochromatic $(k + 1)$-AP, then the desired outcome would have been achieved. So, assume every block contains a k-term arithmetic progression r-color-focus without a monochromatic $(k + 1)$-term AP.

Now, we label every element in the set $[1, N]$ by $b_{x,y}$ where x indicates the index of the block the element is in, and y the location of the element in their blocks. For example, we denote the elements of the k-AP r-color-focus in the block B_a by:

$$P_{a,1} = b_{a,\alpha}, b_{a,\alpha+\beta}, \cdots, b_{a,\alpha+(k-1)\beta}, f_a$$

$$P_{a,2} = b_{a,\gamma}, b_{a,\gamma+\delta}, \cdots, b_{a,\gamma+(k-1)\delta}, f_a$$

$$\vdots$$

$$P_{a,r} = b_{a,\varphi}, b_{a,\varphi+\phi}, \cdots, b_{a,\varphi+(k-1)\phi}, f_a$$

where these r k-term AP converges to f_a, and $P_{x,y}$ denotes the y^{th} progression in the x^{th} block. Since all the k blocks $B_a, B_{a+d}, \cdots, B_{a+(k-1)}$ are identically colored, then there exists these k-term arithmetic progressions:

$$P_{a,1} = b_{a,\alpha}, b_{a,\alpha+\beta}, \cdots, b_{a,\alpha+(k-1)\beta}, f_a$$

$$P_{a+d,1} = b_{a+d,\alpha}, b_{a+d,\alpha+\beta}, \cdots, b_{a+d,\alpha+(k-1)\beta}, f_{a+d}$$

$$\vdots$$

$$P_{a+(k-1)d,1} = b_{a+(k-1)d,\alpha}, b_{a+(k-1)d,\alpha+\beta}, \cdots, b_{a+(k-1)d,\alpha+(k-1)\beta}, f_{a+(k-1)d}$$

$$P_{a,2} = b_{a,\gamma}, b_{a,\gamma+\delta}, \cdots, b_{a,\gamma+(k-1)\delta}, f_a$$

$$\vdots$$

$$P_{a+(k-1)d,2} = b_{a+(k-1)d,\gamma}, b_{a+(k-1)d,\gamma+\delta}, \cdots, b_{a+(k-1)d,\gamma+(k-1)\delta}, f_{a+(k-1)d}$$

$$\vdots$$

$$P_{a+(k-1)d,r} = b_{a+(k-1)d,\varphi}, b_{a+(k-1)d,\varphi+\phi}, \cdots, b_{a+(k-1)d,\varphi+(k-1)\phi}, f_{a+(k-1)d}$$

such that $\chi(P_{a,i}) = \chi(P_{a+d,i}) = \cdots = \chi(P_{a+(k-1)d,i})$ for all $i \in [1, r]$, where χ denotes the coloring configuration of each progression. Each of the r k-term arithmetic progression in all the k blocks will form our $(r + 1)$-color-focus. Consider these k-term arithmetic progressions:

$$F_1 = b_{a,\alpha}, b_{a+d,\alpha+\beta}, \cdots, b_{a+(k-1)d,\alpha+(k-1)\beta}$$

$$F_2 = b_{a,\gamma}, b_{a+d,\gamma+\delta}, \cdots, b_{a+(k-1)d,\gamma+(k-1)\delta}$$

$$\vdots$$

$$F_r = b_{a,\varphi}, b_{a+d,\varphi+\phi}, \cdots, b_{a+(k-1)d,\varphi+(k-1)\phi}$$

$$F_{r+1} = f_a, f_{a+d}, \cdots, f_{(a+(k-1)d)}$$

Each of the progressions F_i are monochromatic of different colors, and are taken from the progression $P_{j,i}$ for all $j \in [a, a + (k - 1)d]$, so every F_i converges to the same point $b_{a+kd,\alpha+k\beta} = b_{a+kd,\gamma+k\delta} = \cdots = b_{a+kd,\varphi+k\phi} = f_{a+kd}$.

In our assumption, $[1, N]$ does not contain any monochromatic $(k + 1)$-term AP, so each of the block color focuses f_i must be of a different color than the r colors used to color r monochromatic k-term APs in each block. Thus, by the definition of a color focus, the progressions $F_1, F_2, \cdots, F_r, F_{r+1}$ forms an $(r + 1)$-color-focus at f_{a+kd}. It completes our claim and the number $W(r + 1, k, l)$ exists.

Now, we have that the number $W(r, k, l)$ exists for all $r \leq l$. When $r = l$, it means we have an l-color-focus on the set $[1, W(r, k, l)]$ where we have l monochromatic k-term APs that converges to a point in the set. Since every monochromatic k-APs are colored differently, the color-focused point must be colored by one of the colors of the k-APs, thus forming a monochromatic $(k+1)$-term arithmetic progression as we desired. Therefore, the number $W(k+1,l)$ exists.

Therefore, the number $W(k,l)$ exists for all k,l and this completes the proof. [7]

4 CONCLUSION

From the 2 main results discussed in this paper, we have seen how the Ramsey Theory exists in the fields of graph theory and numbering sequence. In essence, both results present one pivotal idea of the Ramsey Theory: the existence of order within chaos. Order in the form of a monochromatic clique within an arbitrary coloring of a complete graph in the Ramsey's Theorem, and in the form of a monochromatic arithmetic progression within a random coloring of a sequence in the van der Waerden's Theorem.

Clearly, this field still offers a huge number of problems. The undiscovered Ramsey numbers and their bounds to name one. Or even the existence of the Ramsey Theory principles in other fields of mathematics that we have not realized to exist. It may even be possible to look further to find the ideas of Ramsey Theory outside mathematics, around us in the universe.

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